BOUNDING DIAMETER OF CONICAL KÄHLER METRIC

YAN LI

Abstract. In this paper we research the differential geometric and algebro-geometric properties of the noncollasping limit in the conical continuity equation which generalize the theory in [17].

1. Introduction

The Ricci flow proposed by Hamilton in [12] has been one of the most powerful tools in geometric analysis with the solution of Poincaré conjecture. Similarly, the Kähler Ricci flow has also become an fundamental tool in the study of Kähler geometry for many years. J. Song, G. Tian and their collaborators ([31], [22], [25], [26], [27], [23], [24], [30]) developed the Analytic Minimal Model Program through Kähler Ricci flow. However, in studying the singularity formation of the Kahler-Ricci flow there are some difficulties because we do not know how to control the lower bound for the Ricci curvature along the flow. To overcome these difficulties, in [16], G. La Nave and G. Tian introduced a new continuity equation. In [17], G. La Nave, G. Tian and Z. L. Zhang investigated the differential geometric and algebro-geometric properties of the noncollapsing limit in the continuity method. Properties of the continuity equation in [16] are very similar with properties of the Kähler Ricci flow.

On the other hand, conical Kähler-Einstein metric plays an essential role in recent great progress about Yau-Tian-Donaldson conjecture, see [4] [5] [6] [29]. In [2] and [11], H. Guenancia and M. Páun constructed smooth approximation of conical metric. Recently, L. M. Shen in [19] got a result that is about maximal time existence of unnormalized conical Kähler Ricci flow. Therefore, a natural problem is that what properties the conical version of continuity equations have.

In this paper we generalize the theory in [17] to the conical version and research the differential and algebro-geometric properties of the limit in the conical continuity equation. We will focus on the noncollasping case.

To begin with, we assume that M is a projective manifold with a Kähler metric $\omega_0 \in c_1(L^{'})$, where $L^{'}$ is a line bundle on M. Let D be a smooth hypersurface in M and $\beta \in (0,1)$. We consider the 1-parameter family of equations:

$$\omega = \omega_0 - t(Ric(\omega) - (1-\beta)[D]), \tag{1.1}$$

where $[D]$ is the current of integration along D. Clearly, the Kähler classes vary according to the linear relation: $[\omega] = [\omega_0] - t(c_1(M) - (1-\beta)c_1(L_D))$, where $[\omega]$ denotes the Kähler class of ω and $c_1(L_D)$ denotes the first Chern class of line bundle L_D associated with hypersurface D.

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Our first theorem is:

Theorem 1.2. *For any initial Kähler metric ω_0, there is a unique singular family of solution ω_t for (1.1) on $M \times [0, T)$,where*

$$T = sup\{t|[\omega_0] - t(c_1(M) - (1-\beta)c_1(L_D)) > 0\}. \tag{1.3}$$

and each ω_t is a conical metric.

If $T < \infty$, we need to examine the limit of ω_t as t tends to T. We have the following result if ω_t is noncollapsing.

Theorem 1.4. *Assume that $([\omega_0] - T(c_1(M) - (1-\beta)c_1(L_D)))^n > 0$, where $n = dim_{\mathbb{C}} M$, then ω_t converge to a unique weakly Kähler metric ω_T such that ω_T is smooth on $M \backslash (\mathcal{S}_M \cup D)$ and satisfies:*

$$\omega_T = \omega_0 - TRic(\omega_T), \text{ on } M \backslash (\mathcal{S}_M \cup D),$$

where

$$\mathcal{S}_M = \bigcap \{F | F \; is\; a\; disivior\; satisfying\; [\omega_0] - T(c_1(M) - (1-\beta)c_1(L_D)) - \rho[F] > 0 \; for\; some\; \rho > 0\}.$$

In [17], the limit space which (M, ω_t) converge to in the Gromov-Hausdorff topology has more regular properties, such as metric structure, algebraic structure. In the conical situation, we also have similar properties.

Theorem 1.5. *Assume as in above theorem, $\beta \in \mathbb{Q} \cap (0, 1)$ and $c_1(L_D)$ is semi-positive, then*

(1) (M, ω_t) converges in the Gromov-Hausdorff topology to a compact path metric space (M_T, d_T) which is the metric completion $(M \backslash (\mathcal{S}_M \cup D), \omega_T)$;

(2) M_T has regular part and singular part, i.e. $M_T = \mathcal{R} \cup \mathcal{S}$, a point $x \in \mathcal{R}$ if and only if the tangent cone at x is $\mathbb{C}^n$;

(3) $\mathcal{S}$ is closed and has real codimension ≥ 2 and $\mathcal{R}$ is geodesically convex;

(4) M_T is homeomorphic to a normal projective variety with $\mathcal{S}$ corresponding to a subvariety.

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2. Existence and uniqueness of conical continuity equation

2.1. **Proof of Theorem 1.2.** First we reduce (1.1) to a scalar equation. Choose a real closed (1, 1) form ψ representing $c_1(M)$ and a smooth volume form Ω such that $Ric(\Omega) = \psi$. Let L_D be a line bundle with a Hermitian metric h_D and s_D be a defining section of L_D. Θ_{h_D} represents the curvature of L_D. By Poincaré-Lelong formula, we have

$$\sqrt{-1}\partial\overline{\partial} \log |s_D|^2_{h_D} = -\Theta_{h_D} + [D].$$

Set $\widetilde{\omega}_t = \omega_0 - t(\psi - (1-\beta)\Theta_{h_D})$ for $t \in [0,T)$. One can easily show that $\omega_t = \widetilde{\omega}_t + t\sqrt{-1}\partial\overline{\partial}u$ satisfies (1.1) if u satisfies

$$(\widetilde{\omega}_t + t\sqrt{-1}\partial\overline{\partial}u)^n = e^u \frac{\Omega}{|s_D|_{h_D}^{2(1-\beta)}}, \tag{2.1}$$

where $\widetilde{\omega}_t + t\sqrt{-1}\partial\overline{\partial}u > 0$.

Proposition 2.2. *The equation (2.1) is solvable for each $t \in [0,T)$.*

This proposition is obvious according to Theorem A in [11].

In [11], H. Guenancia and M. Páun introduced that for any $\epsilon > 0$, the function $\chi_\beta : [\epsilon^2, \infty) \to \mathbb{R}$ defined as follows:

$$\chi_\beta(\epsilon^2 + t) = \beta \int_0^r \frac{(\epsilon^2 + r)^\beta}{r}$$

for any $t \geq 0$. There exists a constant C such that $0 \leq \chi_\beta(t) \leq C$ provided that t belongs to a bounded interval. This function is useful to prove Theorem (1.4).

To prove the uniqueness we argue as Jeffery [13].

Proposition 2.3. *Assume that u is a solution of equation(2.1) such that $\omega_t = \widetilde{\omega}_t + t\sqrt{-1}\partial\overline{\partial}u$ is a conical metric. Then u is unique.*

Proof. Assume u_1 and u_2 are solutions of equation(2.1). Set $v = u_1 - u_2$. One immediately obtains the following equation.

$$e^v \omega_1^n = (\omega_1 + t\sqrt{-1}\partial\overline{\partial}v)^n,$$

where $\omega_1 = \widetilde{\omega}_t + t\sqrt{-1}\partial\overline{\partial}u_1$. Let $F_k = \frac{1}{k}|s_D|_{h_D}^{2p}(2p < \beta)$ and $v_k = v + F_k$. It is easy to show

$$\sqrt{-1}\partial\overline{\partial}|s_D|_{h_D}^{2p} \geq -p|s_D|_{h_D}^{2p}\Theta_{h_D}.$$

For each k, v_k can attain maximum at $P_k \in M\backslash D$. Then at P_k one knows

$$e^v(\det g^1_{i\bar{j}}) = \det(g^1_{i\bar{j}} + \sqrt{-1}\partial_i\overline{\partial}_{\bar{j}}(v_k - F_k)).$$

Choose normal coordinate at P_k which simultaneously diagonalize $(g^1_{i\bar{j}})$ and $(\sqrt{-1}\partial_i\overline{\partial}_{\bar{j}}(v_k - F_k))$, i.e. $g^1_{i\bar{j}}(P_k) = \delta_{ij}$ and $\sqrt{-1}\partial_i\overline{\partial}_{\bar{j}}(v_k - F_k)(P_k) = \delta_{ij}((v_k)_{i\bar{j}} - (F_k)_{i\bar{j}})$. Notice that $(v_k)_{i\bar{i}}(P_k) \leq 0$. Then one has

$$\begin{aligned}
e^{v(P_k)} &= \prod_{i=1}^k (1 + (v_k)_{i\bar{i}} - (F_k)_{i\bar{i}}) \\
&\leq \prod_{i=1}^k (1 - (F_k)_{i\bar{i}}) \leq \prod_{i=1}^k (1 + \frac{C}{k}(\Theta_{h_D})_{i\bar{i}}) \\
&\leq \prod_{i=1}^k (1 + \frac{A}{k})^n.
\end{aligned}$$

Let $k \to \infty$, one obtains

$$v \leq 0$$

By the similar argument, one has $v \geq 0$. Therefore, $u_1 = u_2$. □

2.2. **Proof of Theorem 1.4.** In this subsection we investigate the regular properties of limit metric.

Lemma 2.4. *Let F be a divisor in a projective manifold M. If F is big, then there is an effective divisor E such that $[F] - \epsilon[E] > 0$ for all sufficiently small $\epsilon > 0$.*

By the assumption of Theorem(1.4) one knows that $[\omega_0]-T(c_1(M)-(1-\beta)c_1(L_D))$ is big, then by the above Lemma there is a effective divisor E such that$[\omega_0]-T(c_1(M)-(1-\beta)c_1(L_D))-\iota[L_E]$ is ample for some ι. Let h_E be a Hermitian metric on L_E and σ_E be a defining section of E. Thus by the ampleness of $[\omega_0] - T(c_1(M) - (1-\beta)c_1(L_D)) - \iota[L_E]$, one knows

$$\widetilde{\omega_T} - \iota Ric(h_E) > 0.$$

Now we begin to prove Theorem(1.4).

Proof. Let $\widetilde{\omega_{t,E}} = \widetilde{\omega}_t + \iota\sqrt{-1}\partial\overline{\partial}\log|\sigma_E|^2_{h_E}$. If $\bar{t}$ is sufficiently small, $\widetilde{\omega_{t,E}}$ is a smooth Kähler metric on $M\backslash E$ for each $t \in [T-\bar{t}, T+\bar{t}]$. Set $\psi_\epsilon = \delta\chi(\epsilon^2+|s_D|^2_{h_D})$ and $\widetilde{\omega_{t,E,\epsilon}} = \widetilde{\omega_{t,E}}+\sqrt{-1}\partial\overline{\partial}\psi_\epsilon$. If δ is sufficiently small, $\widetilde{\omega_{t,E,\epsilon}}$ is also a smooth Kähler metric on $M\backslash E$ for all ϵ and $t \in [T-\bar{t}, T]$. Now we consider the following approximation equation

$$(\widetilde{\omega_{t,E,\epsilon}} + \sqrt{-1}\partial\overline{\partial}(tv_\epsilon - \iota\log|\sigma_E|^2_{h_E}))^n = e^{\frac{\psi_\epsilon}{t}+v_\epsilon}\frac{\Omega}{(\epsilon^2+|s_D|^2_{h_D})^{1-\beta}}.$$

Set $w_\epsilon = tv_\epsilon - \iota\log|\sigma_E|^2_{h_E}$. Assume that w_ϵ attains minimum at y_0, one has

$$\frac{\psi_\epsilon}{t} + \frac{1}{t}(w_\epsilon + \iota\log|\sigma_E|^2_{h_E}) \geq \log\frac{(\epsilon^2+|s_D|^2_{h_D})^{1-\beta}(\widetilde{\omega_{t,E,\epsilon}})^n}{\Omega} \geq -C.$$

Therefore

$$w_\epsilon \geq -C - \iota\log|\sigma_E|^2_{h_E} \geq -C.$$

For the upper bound of w_ϵ, one needs to consider the following equation. For $t \in [T-\bar{t}, T)$

$$(\widetilde{\omega}_t + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon + t\sqrt{-1}\partial\overline{\partial}v_\epsilon)^n = e^{\frac{\psi_\epsilon}{t}+v_\epsilon}\frac{\Omega}{(\epsilon^2+|s_D|^2_{h_D})^{1-\beta}}.$$

Although $\widetilde{\omega}_t + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon$ may not be a Kähler metric near T, it can be controlled from above, we still make use of maximum principle to get

$$\sup v_\epsilon \leq C.$$

Combining above consequences, one obtains

$$-C \leq w_\epsilon \leq C - \iota\log|\sigma_E|^2_{h_E}.$$

Set $\omega_{t,E,\epsilon} = \widetilde{\omega_{t,E,\epsilon}} + \sqrt{-1}\partial\overline{\partial}w_\epsilon$ and $t_0 = T - \bar{t}$.

Claim 2.5. For $t \in [t_0, T]$, there exist two constants C and α which are independent of t and ϵ such that

$$C^{-1}|\sigma_E|_{h_E}^{2\alpha(n-1)+\frac{2\iota}{t}}\widetilde{\omega_{t_0,E,\epsilon}} \leq \omega_{t,E,\epsilon} \leq \frac{C}{|\sigma_E|_{h_E}^{2\alpha}}\widetilde{\omega_{t_0,E,\epsilon}}.$$

Proof. Set $F_{t_0,D,\epsilon} = \log \frac{\Omega}{(\epsilon^2+|s_D|_{h_D}^2)^{1-\beta}(\widetilde{\omega_{t_0,E,\epsilon}})^n}$. By Yau's Schwarz lemma [32], one deduces

$$\triangle_{\omega_{t,E,\epsilon}} \log tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon} \geq \frac{1}{tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}}(-g^{i\bar{j}}(\widetilde{\omega_{t_0,E,\epsilon}})R_{i\bar{j}}(\omega_{t,E,\epsilon})+g^{i\bar{j}}(\omega_{t,E,\epsilon})g_{k\bar{l}}(\omega_{t,E,\epsilon})R_{i\bar{j}}{}^{k\bar{l}}(\widetilde{\omega_{t_0,E,\epsilon}})).$$

Now we take an holomorphic orthonormal coordinates at a point (t,p) such that $g_{i\bar{j}}(\widetilde{\omega_{t_0,E,\epsilon}}) = \delta_{ij}$, and $g_{i\bar{j}}(\omega_{t,E,\epsilon}) = \lambda_i\delta_{ij}$. So we have

$$-g^{i\bar{j}}(\widetilde{\omega_{t_0,E,\epsilon}})R_{i\bar{j}}(\omega_{t,E,\epsilon}) = \triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}(\frac{\psi_\epsilon}{t}+\frac{1}{t}(w_\epsilon+\iota\cdot\log|\sigma_E|_{h_E}^2)+F_{t_0,D,\epsilon}) - \sum_{i,k}R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}),$$

and

$$g^{i\bar{j}}(\omega_{t,E,\epsilon})g_{k\bar{l}}(\omega_{t,E,\epsilon})R_{i\bar{j}}{}^{k\bar{l}}(\widetilde{\omega_{t_0,E,\epsilon}}) = \sum_{i,k}\frac{\lambda_k}{\lambda_i}R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}).$$

Thus we have

$$\triangle_{\omega_{t,E,\epsilon}} \log tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon} \geq \frac{1}{\sum_m \lambda_m}\{\sum_{i<k}(\frac{\lambda_i}{\lambda_k}+\frac{\lambda_k}{\lambda_i}-2)R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}) + \triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}(\frac{\psi_\epsilon}{t}+\frac{1}{t}(w_\epsilon+\iota\cdot\log|\sigma_E|_{h_E}^2)+F_{t_0,D,\epsilon})\} \tag{2.6}$$

The following result is contained in [11]. We denote $\Psi_{\epsilon,\rho} = C\chi_\rho(\epsilon^2+|s_D|_{h_D}^2)$ and there exist constants C and $0<\rho<1$ such that

$$R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}) \geq -(C+(\Psi_{\epsilon,\rho})_{i\bar{i}}).$$

Using the symmetry of the curvature tensor, we also have

$$R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}) \geq -(C+(\Psi_{\epsilon,\rho})_{k\bar{k}}).$$

Notice that

$$\frac{1}{\sum_m \lambda_m}\sum_{i<k}(\frac{\lambda_i}{\lambda_k}+\frac{\lambda_k}{\lambda_i}-2)R_{i\bar{i}k\bar{k}}(\widetilde{\omega_{t_0,E,\epsilon}}) \geq -\frac{1}{\sum_m \lambda_m}\sum_{i<k}\{\frac{\lambda_k}{\lambda_i}(C+(\Psi_{\epsilon,\rho})_{i\bar{i}})+\frac{\lambda_i}{\lambda_k}(C+(\Psi_{\epsilon,\rho})_{k\bar{k}})\}$$

and

$$\triangle_{\omega_{t,E,\epsilon}}(\Psi_{\epsilon,\rho}) = \sum_i\frac{(\Psi_{\epsilon,\rho})_{i\bar{i}}}{\lambda_i} \geq \frac{1}{\sum_m \lambda_m}\sum_{i<k}\{\frac{\lambda_k}{\lambda_i}(C+(\Psi_{\epsilon,\rho})_{i\bar{i}})+\frac{\lambda_i}{\lambda_k}(C+(\Psi_{\epsilon,\rho})_{k\bar{k}})\} - Ctr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}}.$$

Therefore one gets

$$\triangle_{\omega_{t,E,\epsilon}} \log(tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}+\Psi_{\epsilon,\rho}) \geq \frac{1}{tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}}\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}(\frac{\psi_\epsilon}{t}+\frac{1}{t}(w_\epsilon+\iota\cdot\log|\sigma_E|_{h_E}^2)+F_{t_0,D,\epsilon}) - Ctr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}}. \tag{2.7}$$

From [11] one knows

$$\sqrt{-1}\partial\bar{\partial}F_{t_0.D,\epsilon} \geq -(C\widetilde{\omega_{t_0,E,\epsilon}} + \sqrt{-1}\partial\bar{\partial}\Psi_{\epsilon,\rho});\ |F_{t_0,D,\epsilon}|_{C^0} \leq C.$$

By taking the trace with respect to $\widetilde{\omega_{t_0,E,\epsilon}}$, we get

$$\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}F_{t_0.D,\epsilon} \geq -nC - \triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}\Psi_{\epsilon,\rho}.$$

Taking a simple calculation, one has

$$\begin{aligned}\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}\Psi_{\epsilon,\rho} = \sum_i \frac{(\Psi_{\epsilon,\rho})_{i\bar{i}}}{\lambda_i} &\geq \frac{\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}(\Psi_{\epsilon,\rho}) + nC}{tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}} - Ctr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}} \\ &\geq -\frac{\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}F_{t_0.D,\epsilon}}{tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}} - Ctr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}}.\end{aligned}$$

Note that

$$\triangle_{\widetilde{\omega_{t_0,E,\epsilon}}}(\frac{\psi_\epsilon}{t} + \frac{1}{t}(w_\epsilon + \iota\cdot\log|\sigma_E|^2_{h_E}) + F_{t_0,D,\epsilon}) = \frac{1}{t}tr_{\widetilde{\omega_{t_0,E,\epsilon}}}(\omega_{t,E,\epsilon} - \widetilde{\omega}_t) \geq -\frac{1}{t}tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\widetilde{\omega}_t \geq -C,$$

the last inequality bases on $\widetilde{\omega_{t_0,E,\epsilon}} \geq C^{-1}\widetilde{\omega}_t$. There is an easy fact that is

$$(tr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}})(tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}) \geq n.$$

Notice that there exists a constant C' such that $\widetilde{\omega_{t,E,\epsilon}} \geq C'\widetilde{\omega_{t_0,E,\epsilon}}$.

We denote $H = \log(tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon} + 2\Psi_{\epsilon,\rho}) - \frac{(1+C)}{C'}w_\epsilon$, then by calculation one has

$$\begin{aligned}\triangle_{\omega_{t,E,\epsilon}}H &\geq -Ctr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}} - \frac{(1+C)}{C'}tr_{\omega_{t,E,\epsilon}}(\omega_{t,E,\epsilon} - \widetilde{\omega_{t,E,\epsilon}}) \\ &\geq tr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}} - n(1+C).\end{aligned}$$

Assume H attains maximum at x_0, one deduces

$$tr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}}(x_0) \leq C.$$

Notice that

$$tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}(x_0) \leq (tr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}}(x_0))^{n-1}\cdot e^{\frac{\psi_\epsilon}{t}+\frac{1}{t}(w_\epsilon+\iota\cdot\log|\sigma_E|^2_{h_E})+F_{t_0,D,\epsilon}}(x_0) \leq C.$$

Therefore according to the estimate of w_ϵ and the boundness of $\Psi_{\epsilon,\rho}$, there exist constants C and α such that

$$\begin{aligned}\log tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon} &\leq \log tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon}(x_0) - \frac{(1+C)}{C'}w_\epsilon(x_0) + \frac{(1+C)}{C'}w_\epsilon + C \\ &\leq C - \alpha\log|\sigma_E|^2_{h_E}.\end{aligned}$$

Furthermore one gets

$$tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon} \leq \frac{C}{|\sigma_E|^{2\alpha}_{h_E}}.$$

By the similar argument one has

$$tr_{\omega_{t,E,\epsilon}}\widetilde{\omega_{t_0,E,\epsilon}} \le (tr_{\widetilde{\omega_{t_0,E,\epsilon}}}\omega_{t,E,\epsilon})^{n-1} \cdot e^{-(\frac{\psi_\epsilon}{t}+\frac{1}{t}(w_\epsilon+\iota\cdot\log|\sigma_E|^2_{h_E})+F_{t_0,D,\epsilon})} \le \frac{C}{|\sigma_E|_{h_E}^{2\alpha(n-1)+\frac{2\iota}{t}}}.$$

□

By the above Claim, one knows that for any compact subset $K \subset M\backslash(D\cup E)$, there exists a constant $C_K > 0$ independent of ϵ and t such that $C_K^{-1}\omega_0 \le \omega_{t,E,\epsilon} \le C_K\omega_0$, i.e. $|\triangle_{\omega_0}w_\epsilon| \le C$. By theorem 17.14 in [10], we have that $|w_\epsilon|_{C^{2,\alpha}} \le C_K^{'}$ on $K\times[T-\bar{t},T]$. Furthermore, by the standard bootstrapping argument one has that for any $l>0$, $|w_\epsilon|_{C^{l,\alpha}} \le C_{K,l}$ on $K\times[T-\bar{t},T]$. By the standard diagonal argument and passing to a subsequence, we see that w_{ϵ_i,t_i} C^∞ converges to a $(1,1)$ form on each compact subset $K\subset M\backslash(D\cup E)$ when $\epsilon_i\to 0$ and $t_i\to T$. Back to equation(2.1), we know that there exists a subsequence $\{t_i\}_{i=1}^\infty$ such that u_{t_i} C^∞ converges to u_T on each compact subset $K\subset M\backslash(D\cup E)$ when $t_i\to T$. A priori, this limit may not be unique. So we still need to prove that u_T is unique, i.e., independent of the subsequence $\{t_i\}_{i=1}^\infty$.

Differentiating t at both sides of equation (2.1), one has

$$\triangle_{\omega_t}\dot{u_t} = \frac{1}{t}\dot{u_t} - \frac{n}{t^2} + \frac{1}{t^2}tr_{\omega_t}\omega_0.$$

By the simple calculation, one gets

$$\triangle_{\omega_t}(u_t - n\log t)^{'} \ge \frac{1}{t}(u_t - n\log t)^{'}.$$

By maximum principle one knows

$$\frac{1}{t}(u_t - n\log t)^{'} \le 0.$$

i.e. $(u_t - n\log t)$ is decreasing as $t\to T$. Combining the previous argument, we see that u_T is unique. Therefore Theorem(1.4) is proved. □

2.3. **Smooth approximation of metric with conical singularities.** In this subsection, we assume L_D is a semi-positive line bundle, i.e. there exists a Hermitian metric h_D such that the curvature $\Theta_{h_D}\ge 0$. Fix $t\in[T-\bar{t},T)$, we consider the approximation equation

$$(\omega_{t,\epsilon})^n = e^{\psi_\epsilon+v_\epsilon}\frac{\Omega}{(\epsilon^2+|s_D|^2_{h_D})^{1-\beta}}, \tag{2.8}$$

where $\omega_{t,\epsilon} = \widetilde{\omega_t} + t\sqrt{-1}\partial\overline{\partial}\psi_\epsilon + t\sqrt{-1}\partial\overline{\partial}v_\epsilon$. By the calculation, one has

$$\begin{aligned} Ric(\omega_{t,\epsilon}) &= -\frac{1}{t}(\omega_{t,\epsilon}-\widetilde{\omega_t}) + Ric(\Omega) + (1-\beta)\sqrt{-1}\partial\overline{\partial}\log(\epsilon^2+|s_D|^2_{h_D}) \\ &= -\frac{1}{t}\omega_{t,\epsilon} + \frac{1}{t}\omega_0 + (1-\beta)\frac{\epsilon^2\langle\nabla s,\overline{\nabla s}\rangle}{(\epsilon^2+|s_D|^2_{h_D})^2} + (1-\beta)\frac{\epsilon^2}{\epsilon^2+|s_D|^2_{h_D}}\Theta_{h_D} \\ &\ge -\frac{1}{t}\omega_{t,\epsilon}. \end{aligned}$$

For fixed $t \in [T-\bar{t}, T)$, by Claim(2.3) we know

$$C_t^{-1}\omega_0 \le A_t^{-1}\widetilde{\omega_{t,\epsilon}} \le \omega_{t,\epsilon} \le A_t\widetilde{\omega_{t,\epsilon}} \le \frac{C_t\omega_0}{|s_D|_{h_D}^{2(1-\beta)}}.$$

Therefore

$$diam(\omega_{t,\epsilon}) \le C_t.$$

Proposition 2.9. *(M, ω_t) is the Gromov-Hausdorff limit of $(M, \omega_{t,\epsilon})$ as $\epsilon \to 0$.*

The proof of the above proposition is the same as the proposition(2.5) of [4], so we omit it.

3. A prior estimate to the conical continuity equation

In this section, we present some estimate to the conical continuity equation (1.1). First, we assume $\beta \in \mathbb{Q}$ and L_D is a semi-positive line bundle. The rationality theorem of Kawamata [14] says that $T \in \mathbb{Q}$. Take a positive integer l_0 such that $Tl_0 \in \mathbb{Z}, Tl_0(1-\beta) \in \mathbb{Z}$ and define the limit line bundle $L = l_0(L' + TK_M + T(1-\beta)L_D)$.

Since the limit class $L' + TK_M + T(1-\beta)L_D$ is nef and big, according to the base point free theorem [15], we may assume l_0 is chosen such that L has no base points. A basis of $H^0(M, L)$ gives a holomorphic map

$$\Phi : M \longrightarrow \mathbb{C}P^N$$

where $N = dimH^0(M, L) - 1$. Let M_{reg} be the set of regular points of Φ. Denote by ω_{FS} the Fubini-Study metric of $\mathbb{C}P^N$ and $\eta_T = \frac{1}{l_0}\Phi^*\omega_{FS}$.

Let $\omega_{t,\epsilon}$, $t \in [T-\bar{t}, T)$, be a solution to (2.7). By putting $\eta_t = \frac{T-t}{T}\omega_0 + \frac{t}{T}\eta_T$, a family of background metrics, the solution $\omega_{t,\epsilon}$ can be written as

$$\omega_{t,\epsilon} = \eta_t + \sqrt{-1}\partial\bar{\partial}u_{t,\epsilon}.$$

Since $\frac{1}{T}(\omega_0 - \eta_T) \in c_1(M) - (1-\beta)c_1(L_D)$, there is a smooth volume form Ω on M and curvature Θ_{h_D} on L_D such that

$$Ric(\Omega) - (1-\beta)\Theta_{h_D} = \frac{1}{T}(\omega_0 - \eta_T).$$

Now we consider the following equation

$$(\eta_t + \sqrt{-1}\partial\bar{\partial}u_{t,\epsilon})^n = e^{\frac{u_{t,\epsilon}}{t}}\frac{\Omega}{(\epsilon^2 + |s_D|_{h_D}^2)^{1-\beta}} \tag{3.1}$$

Lemma 3.2. *There is a constant C independent of t and ϵ such that*

$$|u_{t,\epsilon}|_{C^0} \le C.$$

Proof. The uniform upper bound of $u_{t,\epsilon}$ is trivial consequence of the maximum principle. The L^∞ bound follows from the capacity calculation of [34] for exactly our case when $u_{t,\epsilon}$ has a uniform upper bound. □

Corollary 3.3. *There exists C independent of t and ϵ such that*

$$C^{-1}\Omega \le \omega_{t,\epsilon}^n \le \frac{C\Omega}{|s_D|_{h_D}^{2(1-\beta)}}.$$

Lemma 3.4. *There exists C independent of t and ϵ such that*

$$\dot{u_{t,\epsilon}} \leq C; \ddot{u_{t,\epsilon}} \leq C.$$

Proof. Differentiating t at both sides of (3.1), one gets

$$tr_{\omega_{t,\epsilon}}\omega_{t,\epsilon}^{'} = \frac{1}{t^2}(t\dot{u_{t,\epsilon}} - u_{t,\epsilon})$$

where

$$\omega_{t,\epsilon}^{'} = \frac{1}{T}(\eta_T - \omega_0) + \sqrt{-1}\partial\overline{\partial}\dot{u_{t,\epsilon}} = \frac{1}{t}(\omega_{t,\epsilon} - \omega_0 - \sqrt{-1}\partial\overline{\partial}u_{t,\epsilon}) + \sqrt{-1}\partial\overline{\partial}\dot{u_{t,\epsilon}}.$$

By the simple calculation one has

$$\triangle_{\omega_{t,\epsilon}}(t\dot{u_{t,\epsilon}} - u_{t,\epsilon}) = \frac{1}{t}(t\dot{u_{t,\epsilon}} - u_{t,\epsilon}) - n + tr_{\omega_{t,\epsilon}}\omega_0.$$

Applying the maximum principle one derives

$$t\dot{u_{t,\epsilon}} - u_{t,\epsilon} \leq C.$$

Combining with the C^0 bound of $u_{t,\epsilon}$ we also have

$$\dot{u_{t,\epsilon}} \leq C.$$

To get the upper bound of $\ddot{u_{t,\epsilon}}$ we first observe that

$$t\dot{u_{t,\epsilon}} - u_{t,\epsilon} = t^2 tr_{\omega_{t,\epsilon}}\omega_{t,\epsilon}^{'}.$$

Differentiating t at both sides of the above formula one gets

$$t\ddot{u_{t,\epsilon}} = 2t \cdot tr_{\omega_{t,\epsilon}}\omega_{t,\epsilon}^{'} + t^2\triangle_{\omega_{t,\epsilon}}\ddot{u_{t,\epsilon}} - t^2|\omega_{t,\epsilon}^{'}|^2 = t^2\triangle_{\omega_{t,\epsilon}}\ddot{u_{t,\epsilon}} - |\omega_{t,\epsilon} - t\omega_{t,\epsilon}^{'}|^2 + n.$$

Then by the maximum principle one derives

$$\ddot{u_{t,\epsilon}} \leq C.$$

□

By theorem(1.2), one knows that $u_{t,\epsilon}$ C^∞ converges to u_t on each compact subset $K \subset M \setminus D$ when $\epsilon \to 0$. Furthermore u_t solves the following equation in the current sense

$$(\eta_t + \sqrt{-1}\partial\overline{\partial}u_t)^n = e^{\frac{u_t}{t}}\frac{\Omega}{|s_D|_{h_D}^{2(1-\beta)}}.$$

Lemma 3.5. *The function u_t converges uniformly to a bounded function u_T satisfying*

$$(\eta_T + \sqrt{-1}\partial\overline{\partial}u_T)^n = e^{\frac{u_T}{T}}\frac{\Omega}{|s_D|_{h_D}^{2(1-\beta)}}$$

in the current sense.

Proof. For u_t we observe that

$$t\triangle_{\omega_t}(\frac{u_t}{t})' = (\frac{u_t}{t})' + \frac{1}{t}(tr_{\omega_t}\omega_0 - n) \geq (\frac{u_t}{t})' - \frac{n}{t}.$$

Then one deduces

$$t\triangle_{\omega_t}(\frac{u_t}{t} - n\log t)' \geq (\frac{u_t}{t} - n\log t)'.$$

By the maximum principle one knows that $\frac{u_t}{t} - n\log t$ is monotone decreasing. Consequently, u_t converges uniformly to a unique limit u_T. It is obvious that u_T is smooth outside $M\backslash(\mathcal{S}_M \cup D)$. □

Proposition 3.6. *There exists C independent of t and ϵ such that*

$$\eta_T \leq C\omega_{t,\epsilon}, \forall t \in [T-\bar{t}, T).$$

Proof. By Yau's Schwarz lemma [32] and $Ric(\omega_{t,\epsilon}) \geq -\frac{1}{t}\omega_{t,\epsilon}$,

$$\triangle_{\omega_{t,\epsilon}} \log tr_{\omega_{t,\epsilon}}\eta_T \geq -\frac{n}{t} - ntr_{\omega_{t,\epsilon}\eta_T}.$$

On the other hand, $\eta_t \geq \delta\eta_T$ for some $\delta > 0$ independent of t, so

$$\triangle_{\omega_{t,\epsilon}} u_{t,\epsilon} = n - tr_{\omega_{t,\epsilon}}\eta_t \leq n - \delta tr_{\omega_{t,\epsilon}}\eta_T.$$

Hence

$$\triangle_{\omega_{t,\epsilon}}(\log tr_{\omega_{t,\epsilon}}\eta_T - \frac{2n}{\delta}u_{t,\epsilon}) \geq ntr_{\omega_{t,\epsilon}}\eta_T - \frac{C(n,T)}{\delta}.$$

Let $H = \log tr_{\omega_{t,\epsilon}}\eta_T - \frac{2n}{\delta}u_{t,\epsilon}$. Assume H achieves maximum at x_0, then

$$tr_{\omega_{t,\epsilon}}\eta_T(x_0) \leq C.$$

By the boundness of $u_{t,\epsilon}$, one has

$$tr_{\omega_{t,\epsilon}}\eta_T \leq C.$$

□

Corollary 3.7. *The limit metric ω_T is smooth on $M_{reg}\backslash D$.*

Proof. η_T is smooth on any compact subset $K \subset M_{reg}\backslash D$, so by Lemma(3.5) and proposition(3.7) one knows

$$C^{-1}\eta_T \leq \omega_T \leq C_K\eta_T.$$

In particular, $n + \triangle_{\eta_T}u_T \leq C_K$ on K. Then applying a bootstrap argument we get the higher derivative bound $|u_T|_{C^l(K)} \leq C_{l,K}$. □

Now we define $w_{t,\epsilon} = (T-t)\dot{u_{t,\epsilon}} + u_{t,\epsilon}$ which satisfies

$$\triangle_{\omega_{t,\epsilon}} w_{t,\epsilon} = \frac{1}{t}w_{t,\epsilon} - \frac{T}{t^2}u_{t,\epsilon} + n - tr_{\omega_{t,\epsilon}}\eta_T. \tag{3.8}$$

This can be seen by combining

$$\triangle_{\omega_{t,\epsilon}}\dot{u_{t,\epsilon}} = \frac{1}{t^2}(t\dot{u_{t,\epsilon}} - u_{t,\epsilon}) + \frac{1}{T}tr_{\omega_{t,\epsilon}}(\omega_0 - \eta_T)$$

and

$$\triangle_{\omega_{t,\epsilon}} u_{t,\epsilon} = n - \frac{T-t}{T} tr_{\omega_{t,\epsilon}}\omega_0 - \frac{t}{T} tr_{\omega_{t,\epsilon}}\eta_T.$$

For (3.8) by maximum principle one gets

$$w_{t,\epsilon} \geq -c$$

. Therefore

$$|w_{t,\epsilon}|_{C^0} \leq C, |\triangle_{\omega_{t,\epsilon}} w_{t,\epsilon}|_{C^0} \leq C.$$

Combining with the C^0 bound of $u_{t,\epsilon}$ we also have

$$-\frac{C}{T-t} \leq \dot{u_{t,\epsilon}} \leq C, \ \forall t \in [T-\bar{t}, T).$$

Proposition 3.9. *There exists C independent of t and ϵ such that*

$$|\nabla w_{t,\epsilon}|_{C^0} \leq C, \ \forall t \in [T-\bar{t}, T).$$

In particular, since $\dot{u_{t,\epsilon}}$ converges to a locally bounded function on $M \backslash (\mathcal{S}_M \cup D)$ as $t \to T$ and $\epsilon \to 0$, one has

$$|\nabla u_T|_{C^0} \leq C, \forall t \in [T-\bar{t}, T).$$

Proof. Recall that $Ric(\omega_{t,\epsilon}) \geq -\frac{1}{t}(\omega_{t,\epsilon} - \omega_0)$, so by the Bochner formula,

$$\triangle|\nabla w_{t,\epsilon}|^2 \geq |\nabla\nabla w_{t,\epsilon}|^2 + |\nabla\overline{\nabla} w_{t,\epsilon}|^2 + \nabla_i \triangle w_{t,\epsilon} \cdot \nabla_{\bar{i}} w_{t,\epsilon} + \nabla_{\bar{i}} \triangle w_{t,\epsilon} \cdot \nabla_i w_{t,\epsilon} + \frac{1}{t}(\omega_0 - \omega_{t,\epsilon})_{i\bar{j}} \nabla_i w_{t,\epsilon} \nabla_{\bar{j}} w_{t,\epsilon}$$

where we omit the metric for the convenience. By (3.8) one has

$$\nabla_i \triangle w_{t,\epsilon} \cdot \nabla_{\bar{i}} w_{t,\epsilon} + \nabla_{\bar{i}} \triangle w_{t,\epsilon} \cdot \nabla_i w_{t,\epsilon} = \frac{2}{t}|\nabla w_{t,\epsilon}|^2 - 2Re(\nabla_i tr_{\omega_{t,\epsilon}\eta_T} \cdot \nabla_{\bar{i}} w_{t,\epsilon}) - \frac{2T}{t^2} Re(\nabla_i u_{t,\epsilon} \cdot \nabla_{\bar{i}} w_{t,\epsilon}).$$

So,

$$\triangle|\nabla w_{t,\epsilon}|^2 \geq \frac{1}{2t}|\nabla w_{t,\epsilon}|^2 - 4t|\nabla tr_{\omega_{t,\epsilon}}\eta_T|^2 - \frac{16T^2}{t^3}|\nabla u_{t,\epsilon}|^2.$$

Notice that

$$\begin{aligned}\triangle tr_{\omega_{t,\epsilon}}\eta_T &\geq tr_{\omega_{t,\epsilon}}\eta_T(-\frac{n}{t} - A tr_{\omega_{t,\epsilon}}\eta_T) + \frac{1}{tr_{\omega_{t,\epsilon}}\eta_T}|\nabla tr_{\omega_{t,\epsilon}}\eta_T|^2 \\ &\geq -C + C|\nabla tr_{\omega_{t,\epsilon}}\eta_T|^2,\end{aligned}$$

and

$$\triangle(-u_{t,\epsilon}) = -n + \frac{T-t}{T} tr_{\omega_{t,\epsilon}}\omega_0 + \frac{t}{T} tr_{\omega_{t,\epsilon}}\eta_T \geq \frac{T-t}{T} tr_{\omega_{t,\epsilon}}\omega_0 - C.$$

and

$$\triangle u_{t,\epsilon}^2 = 2u_{t,\epsilon}\triangle u_{t,\epsilon} + 2|\nabla u_{t,\epsilon}|^2 \geq 2|\nabla u_{t,\epsilon}|^2 - C\frac{T-t}{T} tr_{\omega_{t,\epsilon}}\omega_0 - C.$$

Let $H = |\nabla w_{t,\epsilon}|^2 + 4tC tr_{\omega_{t,\epsilon}}\eta_T + \frac{8T^2}{t^3} u_{t,\epsilon}^2 - \frac{8T^2}{t^3} C u_{t,\epsilon}$, then one obtains

$$\triangle H \geq \frac{1}{2t}|\nabla w_{t,\epsilon}|^2 - C,$$

by the maximum principle one gets

$$|\nabla w_{t,\epsilon}|_{C^0} \leq C.$$

$\square$

4. Algebraic structure of the limit space

4.1. **Preliminaries.** In this subsection we introduce some useful formulas on a general line bundle. Let (M,ω) be a Kähler manifold of dimension n and (L,h) be a Hermitian line bundle over M. Let Θ_h be the Chern curvature form of h. Let ∇ and $\overline{\nabla}$ denote the $(1,0)$ and $(0,1)$ part of a connection respectively. The connection appeared in this paper is usually known as the Chern connection or Levi-Civita connection.

For a holomorphic section $\tau \in H^0(M,L)$ we write for simplicity

$$|\tau| = |\tau|_h,\ |\nabla\tau|_{h\otimes\omega} = |\nabla\tau|,$$

and

$$|\nabla\nabla\tau|^2 = \sum_{i,j}|\nabla_i\nabla_j\tau|^2,\ |\nabla\overline{\nabla}\tau|^2 = \sum_{i,j}|\nabla_i\nabla_{\bar{j}}\tau|^2.$$

By direct computation we have

Lemma 4.1. *(Bochner formulas). For any $\tau \in H^0(M,L)$ one has*

$$\triangle_\omega|\tau|^2 = |\nabla\tau|^2 - |\tau|^2\cdot tr_\omega\Theta_h \tag{4.2}$$

and

$$\begin{aligned}\triangle_\omega|\nabla\tau|^2 = |\nabla\nabla\tau|^2 + |\nabla\overline{\nabla}\tau|^2 - \nabla_j(\Theta_h)_{i\bar{j}}\langle\tau,\nabla_{\bar{i}}\bar{\tau}\rangle - \nabla_{\bar{j}}(tr_\omega\Theta_h)\langle\nabla_j\tau,\bar{\tau}\rangle \\ + R_{i\bar{j}}\langle\nabla_j\tau,\nabla_{\bar{i}}\bar{\tau}\rangle - 2(\Theta_h)_{i\bar{j}}\langle\nabla_j\tau,\nabla_{\bar{i}}\bar{\tau}\rangle - |\nabla\tau|^2\cdot tr_\omega\Theta_h\end{aligned} \tag{4.3}$$

where $R_{i\bar{j}}$ is the Ricci curvature of ω, $\langle,\rangle$ is the inner product defined by h.

4.2. **Gromov-Hausdorff convergence: global convergence.** In this subsection we consider a family of manifolds $(M,\omega_{t,\epsilon})$ on which the lower bound of Ricci curvature can be controlled, i.e. $Ric(\omega_{t,\epsilon}) \geq -\frac{1}{T-\bar{t}}\omega_{t,\epsilon}$ for $t \in [T-\bar{t},T)$. By Gromov precompactness theorem, passing to a subsequence $(t_i,\epsilon_i) \to (T,0)$ and fix $x_0 \in M\backslash(\mathcal{S}_M \cup D)$, we may assume that

$$(M,\omega_{t_i,\epsilon_i},x_0) \xrightarrow{d_{GH}} (M_T,d_T,x_T).$$

The limit (M_T,d_T) is a complete length metric space, maybe noncompact in a prior. It has a regular/singular decomposition $M_T = \mathcal{R}\cup\mathcal{S}$, a point $x\in\mathcal{R}$ iff the tangent cone at x is the Euclidean space $\mathbb{R}^{2n}$. The proof of the following lemma is exactly same as [28] so we omit it.

Lemma 4.4. *There is a sufficiently small constant $\delta > 0$ such that for any $t \in [T-\bar{t},T)$ and $\epsilon \geq 0$, if a metric ball $B_{\omega_{t,\epsilon}}(x,r)$ satisfies*

$$Vol(B_{\omega_{t,\epsilon}}(x,r)) \geq (1-\delta)Vol(B^0_r) \text{ and } B_{\omega_{t,\epsilon}}(x,r)\cap D = \varnothing$$

where $Vol(B^0_r)$ is the volume of a metric ball of radius r in $2n$-Euclidean space, then

$$Ric(\omega_{t,\epsilon}) \leq (2n-1)r^{-2}\omega_{t,\epsilon},\ \text{in } B_{\omega_{t,\epsilon}}(x,\delta r).$$

Lemma 4.5. *The regular set $\mathcal{R}$ is open in the limit space (M_T, d_T, x_T).*

Proof. We follow Tian's argument [29]. By Proposition (2.9), one has $(M, \omega_{t_i}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T)$. If $x \in \mathcal{R}$, then by Colding's volume convergence theorem [7], there exists $r = r(x) > 0$ such that $\mathcal{H}^{2n}(B_{d_T}(x, r)) \geq (1 - \frac{\delta}{2})Vol(B^0_r)$, where $\mathcal{H}^{2n}$ denotes the Hausdorff measure. Let $\{x_i\}$ be a sequence of points in M such that $x_i \xrightarrow{d_{GH}} x$, then by the volume convergence theorem again, $Vol(B_{\omega_{t_i}}(x_i, r)) \geq (1-\delta)Vol(B^0_r)$ for i sufficiently large. On the other hand, if $y_i \in D$, then by the Bishop-Gromov volume comparison theorem, for any $\bar{r} > 0$, one has (set $a = -\frac{1}{T-\bar{t}}$)

$$\frac{Vol(B_{\omega_{t_i}}(y_i, \bar{r}))}{Vol(B^a_{\bar{r}})} \leq \beta.$$

Furthermore, when $\bar{r}$ is sufficiently small, we have

$$\frac{Vol(B_{\omega_{t_i}}(y_i, \bar{r}))}{Vol(B^0_{\bar{r}})} = \frac{Vol(B_{\omega_{t_i}}(y_i, \bar{r}))}{Vol(B^a_{\bar{r}})} \cdot \frac{Vol(B^a_{\bar{r}})}{Vol(B^0_{\bar{r}})} \leq (1+\delta)\beta.$$

Note that $B_{\omega_{t_i}}(x_i, r) \xrightarrow{d_{GH}} B_{d_T}(x, r)$, so by the Bishop-Gromov volume comparison theorem there exists an $N = N(\delta)$ such that for any $\tilde{r} \in (0, \frac{r}{N})$ and $y_i \in B_{\omega_{t_i}}(x_i, \tilde{r})$, one gets

$$1 - \delta \leq \frac{Vol(B_{\omega_{t_i}}(y_i, \tilde{r}))}{Vol(B_{\omega_{t_i}}(x_i, \tilde{r}))} \leq 1 + \delta.$$

Now, we claim that $B_{\omega_{t_i}}(x_i, r') \cap D = \varnothing$ where $r' = min\{\bar{r}, \tilde{r}\}$. If this claim is false, we assume $y_i \in B_{\omega_{t_i}}(x_i, r')$ for all i sufficiently large, we have

$$1 - \delta \leq \frac{Vol(B_{\omega_{t_i}}(x_i, r'))}{Vol(B^0_{r'})} \leq (1+\delta)\frac{Vol(B_{\omega_{t_i}}(y_i, r'))}{Vol(B^0_{r'})} \leq (1+\delta)^2\beta.$$

Then we get a contradiction if δ is chosen sufficiently small. According to above lemma, together with Andersons harmonic radius estimate [1], there is $\delta' = \delta'(\alpha) > 0$ for any $0 < \alpha < 1$ such that the $C^{1,\alpha}$ harmonic radius at x_i is bigger than $\delta' r'$. Passing to the limit, it gives a harmonic coordinate on $B_{d_T}(x, \delta' r')$. This implies in particular that $B_{d_T}(x, \delta' r') \subset \mathcal{R}$. So $\mathcal{R}$ is open with a $C^{1,\alpha}$ Kähler metric, denoted by $\overline{\omega_T}$; moreover the metric ω_{t_i,ϵ_i} or ω_{t_i} converges in $C^{1,\alpha}$ topology to $\overline{\omega_T}$ on $\mathcal{R}$ for any $0 < \alpha < 1$. □

For any metric ω, let d_ω be the length metric induced by ω.

Lemma 4.6. $(M_T, d_T) = \overline{(\mathcal{R}, d_{\overline{\omega_T}})}$, *the metric completion of* $(\mathcal{R}, d_{\overline{\omega_T}})$.

Proof. By the previous argument one finds an exhaustion of $\mathcal{R}$ by compact subsets K_i with $K_i \subset K_{i+1}$ and a sequence of embeddings $\phi_i : K_i \to M$ such that $\phi_i(x_T) = x_0$. Thus ϕ_i defines a Gromov-Hausdorff approximation of the convergence $(M, \omega_{t_i,\epsilon_i}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T)$ because $Codim(\mathcal{S}) \geq 2$ [3]. There is a fact that $\phi_i^* \omega_{t_i,\epsilon_i} \xrightarrow{C^{1,\alpha}} \overline{\omega_T}$ which demonstrates that $(\mathcal{R}, d_T|_{\mathcal{R}}) = (\mathcal{R}, d_{\overline{\omega_T}})$. Notice that $(\mathcal{R}, d_T)$ is dense in (M_T, d_T, x_T) because $Codim(\mathcal{S}) \geq 2$. Therefore the lemma is proved. □

Lemma 4.7. *$\mathcal{R}$ is geodesically convex in M_T in the sense that any minimal geodesic with endpoints in $\mathcal{R}$ lies in $\mathcal{R}$.*

Proof. It is simply a consequence of Colding-Nabers Hölder continuity of tangent cones along a geodesic in M_T [8]. Actually, in [3] one knows that any pair of regular points can be connected by a curve consisting entirely of almost regular points and knows $\mathcal{R}$ is locally convex by previous argument. Therefore, the tangent cone of each point which is in a minimal geodesic connecting any pair of regular points is $\mathbb{R}^{2n}$. $\square$

Let D' be any divisor such that $D \cup \mathcal{S}_M \subset D'$. Define the Gromov-Hausdorff limit of D'

$$D'_T := \{x \in M_T | there\ exists\ x_i \in D' such\ that\ x_i \xrightarrow{d_{GH}} x\}.$$

Proposition 4.8. *(M_T, d_T) is isometric to $\overline{(M \backslash D', d_{\omega_T})}$.*

Proof. First, by the argument of [18] one knows that $(M_T \backslash D'_T, \overline{\omega_T})$ is isometric to $(M \backslash D', \omega_T)$; moreover $M_T \backslash D'_T \subset \mathcal{R}$. We make the following

Claim 4.9. $D'_T \backslash \mathcal{S}$ is a subvariety of dimension $(n-1)$ if it is not empty.

Proof. Let $x \in D'_T \backslash \mathcal{S}$ and $x_i \in D'$ such that $x_i \xrightarrow{d_{GH}} x$. By the $C^{1,\alpha}$ convergence of ω_{t_i,ϵ_i} around x, there are $C, r > 0$ independent of i and a sequence of harmonic coordinates in $B_{\omega_{t_i,\epsilon_i}}(x_i, r)$ such that $C^{-1}\omega_E \leq \omega_{t_i,\epsilon_i} \leq C\omega_E$ where ω_E is the Euclidean metric in the coordinates. Since the total volume of D' is uniformly bounded for any ω_{t_i,ϵ_i}, the local analytic $D' \cap B_{\omega_{t_i,\epsilon_i}}(x_i, r)$ have a uniform bound of degree and so converge to an analytic set $D'_T \cap B_{d_T}(x, r)$. $\square$

From the above Claim we know that $dim(D'_T) = dim(\mathcal{S} \cup (D'_T \backslash \mathcal{S})) \leq 2n - 2$. Thus by the argument of [3], one can show that the length metric $d_{\overline{\omega_T}}$ on $M_T \backslash D'_T$ is the same as d_T. Therefore

$$(M_T, d_T) = \overline{(M_T \backslash D'_T, d_{\overline{\omega_T}})} = \overline{(M \backslash D', d_{\omega_T})}.$$

$\square$

Combining with Proposition(2.8), a direct corollary is

Corollary 4.10. *(M, ω_t, x_0) converges globally to (M_T, d_T, x_T) under the Gromov-Hausdorff topology as $t \to T$.*

Let M_{sing} be the subvariety of critical points of Φ which is defined in section 3 and $M_{reg} = M \backslash M_{sing}$. We have shown that ω_T is a smooth metric on $M_{reg} \backslash D$. Another corollary is

Corollary 4.11. *(M_T, d_T) is isometric to $\overline{(M_{reg} \backslash D, d_{\omega_T})}$.*

Proof. We choose a divisor D' such that $D' \supset (D \cup \mathcal{S}_M \cup M_{sing})$, then $M \backslash D' \subset M_{reg} \backslash D$. Notice that $(M_{reg} \backslash D) \backslash (M \backslash D') = (M_{reg} \backslash D) \cap D'$ has real codimension larger than 2 in $(M_{reg} \backslash D, \omega_T)$. Thus the length metric d_{ω_T} on $M \backslash D'$ equals to the restricted extrinsic metric from $(M_{reg} \backslash D, \omega_T)$. Since $M \backslash D'$ is dense in $M_{reg} \backslash D$, we conclude

$$(M_T, d_T) = \overline{(M \backslash D', d_{\omega_T})} = \overline{(M_{reg} \backslash D, d_{\omega_T})}.$$

$\square$

Lemma 4.12. *The identity map* id : $M_{reg}\backslash D \to M$ *gives a Gromov-Hausdorff approximation representing the convergence* $(M, \omega_t, x_0) \to (M_T, d_T, x_T)$ *as* $t \to T$.

Proof. First we observe that $(M\backslash D', d_T) = (M\backslash D', d_{\omega_T})$ and $(M\backslash D', d_T)$ is dense in$(M_{reg}\backslash D, d_T)$. Thus id : $(M\backslash D', d_{\omega_T}) \to (M, \omega_t)$ defines a Gromov-Hausdorff approximation because $(M\backslash D', d_{\omega_T})$ is dense in (M_T, d_T). $\square$

Therefore, the identity map id extends to an isometry

$$\overline{\mathrm{id}}: \ \overline{(M_{reg}\backslash D, d_{\omega_T})} \to (M_T, d_T).$$

Since ω_T is smooth on $M_{reg}\backslash D$, one sees that $M_{reg}\backslash D \subset \mathcal{R}$.

Proposition 4.13. *(1)* $\omega_{t,\epsilon}$ *converges smoothly to* ω_T *on* $M_{reg}\backslash D$ *as* $t \to T$ *and* $\epsilon \to 0$.

(2) $\overline{\mathrm{id}}(M_{reg}\backslash D) = \mathcal{R}$*, the regular set of* M_T.

Proof. (1) For any compact subset $K \subset M_{reg}\backslash D \subset \mathcal{R}$, there exists $r = r_K > 0$ such that $Vol(B_{d_T}(x, r)) \geq (1 - \frac{\delta}{2})Vol(B^0_r)$ for any $x \in K$. where δ is the constant in Lemma (4.4). Then, since the identity map represents the Gromov-Hausdorff convergence, we have $Vol(B_{\omega_{t,\epsilon}}(x, r)) \geq (1 - \delta)Vol(B^0_r)$ for any $x \in K$, t sufficiently close to T and ϵ sufficiently close to 0. By Lemma(4.4), the Ricci curvature $Ric(\omega_{t,\epsilon}) \leq C\omega_{t,\epsilon}$ uniformly on K for some constant $C = C(K)$. Since

$$\omega_{t,\epsilon} = \omega_0 - tRic(\omega_{t,\epsilon}) + (1-\beta)\frac{\epsilon^2\langle \nabla s, \overline{\nabla s}\rangle}{(\epsilon^2 + |s_D|^2_{h_D})^2} + (1-\beta)\frac{\epsilon^2}{\epsilon^2 + |s_D|^2_{h_D}}\Theta_{h_D},$$

one sees that $\omega_{t,\epsilon} \geq C^{-1}\omega_0$. Notice that

$$tr_{\omega_0}\omega_{t,\epsilon} \leq (tr_{\omega_{t,\epsilon}}\omega_0)^{n-1}\frac{(\omega_{t,\epsilon})^n}{\omega_0^n}.$$

Together with the uniform L^∞ bound of $u_{t,\epsilon}$, one gets

$$C^{-1}\omega_0 \leq \omega_{t,\epsilon} \leq C\omega_0, \ \text{on } K.$$

Then by a standard bootstrap argument, we prove that $\omega_{t,\epsilon}$ converges smoothly to ω_T on K.

(2) We only to prove $M_{reg}\backslash D \supset \mathcal{R}$. We argue by contradiction. Suppose there is a point $p \in \mathcal{R}\backslash(M_{reg}\backslash D)$, then there exists a family of points $p_i \in M_{sing} \cup D$ such that $p_i \xrightarrow{d_{GH}} p$. We will divide the discussion into two parts.

On one hand, if there exists $p_i \xrightarrow{d_{GH}} p$ for each $p_i \in D$, that is a contradiction by Lemma (4.5).

On the other hand, there exists $p_i \xrightarrow{d_{GH}} p$ for each $p_i \in M_{sing}\backslash D$. By $C^{1,\alpha}$ convergence on $\mathcal{R}$, there exist $C, r > 0$ independent of t and ϵ and a sequence of harmonic coordinates on $B_{\omega_{t_i,\epsilon_i}}(p_i, r)$ such that $C^{-1}\omega_E \leq \omega_{t_i,\epsilon_i} \leq C\omega_E$ where ω_E is the Euclidean metric in this

coordinate. Denote $m = dim_{\mathbb{C}} M_{sing}$. Then

$$Vol_{\omega_{t_i,\epsilon_i}}((M_{sing}\backslash D)\cap B_{\omega_{t_i,\epsilon_i}}(p_i,r)) = \int_{(M_{sing}\backslash D)\cap B_{\omega_{t_i,\epsilon_i}}(p_i,r)} \omega^m_{t_i,\epsilon_i} \geq \int_{(M_{sing}\backslash D)\cap B_{\omega_E}(C^{-\frac{1}{2}})} (C^{-1}\omega_E)^m$$

which has a uniform lower bound $C^{-2m}c(m)r^{2m}$ where $c(m)$ is the volume of unit sphere in $\mathbb{C}^m$. However, this contradicts with the degeneration of the limit metric η_T along M_{sing}:

$$Vol_{\omega_{t_i,\epsilon_i}}((M_{sing}\backslash D)\cap B_{\omega_{t_i,\epsilon_i}}(p_i,r)) \leq Vol_{\omega_{t_i,\epsilon_i}}(M_{sing}\backslash D) = \int_{M_{sing}\backslash D} \omega^n_{t_i,\epsilon_i} = \left(\frac{T-t_i}{T}\right)^m \int_{M_{sing}\backslash D} \omega_0^m$$

which tends to 0 as $t_i \to T$. So we have $M_{reg}\backslash D \supset \mathcal{R}$. $\square$

4.3. L^∞**estimate to holomorphic sections.** Let $L = l_0(L' + TK_M + T(1-\beta)L_D)$ be the limit line bundle. Choose a Hermitian metric $h_{L'}$ on L' whose curvature form $\Theta_h = \omega_0$ and put $h_{t,\epsilon} = h^{l_0}_{L'} \otimes (\omega^{-n}_{t,\epsilon})^{l_0T} \otimes h_D^{l_0T(1-\beta)} \cdot e^{-l_0T(1-\beta)\log(\epsilon^2+|s_D|^2_{h_D})}$, a family of Hermitian metric on L for any $t \in [T-\bar{t}, T)$. The curvature form of $h_{t,\epsilon}$ is

$$\Theta_{h_{t,\epsilon}} = l_0\frac{T}{t}\omega_{t,\epsilon} - l_0\frac{T-t}{t}\omega_0 \leq l_0\frac{T}{t}\omega_{t,\epsilon}.$$

So, by the Bochner formula (4.2) we have

$$\triangle_{\omega_{t,\epsilon}}|\tau|^2_{h^k_{t,\epsilon}} = |\nabla\tau|^2_{h^k_{t,\epsilon}} - knl_0\frac{T}{t}|\tau|^2_{h^k_{t,\epsilon}},\ \forall\, \tau \in H^0(M, L^k).$$

Also recall that we have the following well-known Sobolev inequality: for any $R > 0$, there is $C(R)$ independent of t and ϵ such that

$$\left(\int_{B_{\omega_{t,\epsilon}}(x_0,R)} |f|^{\frac{2n}{n-1}}\omega^n_{t,\epsilon}\right)^{\frac{n-1}{n}} \leq C(R)\int_{B_{\omega_{t,\epsilon}}(x_0,R)} |f|^2 + |\nabla f|^2_{\omega_{t,\epsilon}}\omega^n_{t,\epsilon}.$$

for all $f \in C^1_0(B_{\omega_{t,\epsilon}}(x_0,R))$.

By a standard iteration argument(Lemma 3.14 [17]) we have

Lemma 4.14. *For any $R > 0$, there exists $C(R)$ independent of t, ϵ and $k \geq 1$ such that for any $t \in [T-\bar{t}, T)$ and $B_{\omega_{t,\epsilon}}(x, 2r) \subset B_{\omega_{t,\epsilon}}(x_0, R)$, if $\tau \in H^0(B_{\omega_{t,\epsilon}}(x, 2r), L^k)$, then*

$$\sup_{B_{\omega_{t,\epsilon}}(x,r)} |\tau|^2_{h^k_{t,\epsilon}} \leq C(R)\cdot r^{-2n}\cdot k^n \cdot \int_{B_{\omega_{t,\epsilon}}(x,2r)} |\tau|^2_{h^k_{t,\epsilon}}\omega^n_{t,\epsilon}.$$

Recall the Gromov-Hausdorff convergence

$$(M, \omega_{t,\epsilon}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T).$$

Define the Hermitian line bundle (L_T, h_T) on the regular set $\mathcal{R} \subset M_T$ by

$$L = l_0(L' + TK_{\mathcal{R}} + T(1-\beta)L_D),\ h_T = h^{l_0}_{L'} \otimes (\omega^{-n}_T)^{l_0T} \otimes h_D^{l_0T(1-\beta)} \cdot e^{-l_0T(1-\beta)\log|s_D|^2_{h_D}}.$$

Under the isometry $\overline{\mathrm{id}} : \overline{(M_{reg}\backslash D, d_{\omega_T})} \to (M_T, d_T)$ and $\mathcal{R} = M_{reg}\backslash D$, we know that the Hermitian line bundles $(L, h_{t,\epsilon})$ converges smoothly to (L_T, h_T) on $\mathcal{R}$ as $t \to T$ and $\epsilon \to 0$.

Corollary 4.15. *Let $R > 0$, $t_i \to T$, $\epsilon \to 0$ and τ_i be a sequence of holomorphic sections of L^k, $k \geq 1$, satisfying*

$$\int_M |\tau_i|^2_{h^k_{h_{t_i,\epsilon_i}}} \omega^n_{t_i,\epsilon_i} \leq 1.$$

Then, passing to a subsequence if necessary, τ_i converges to a locally bounded holomorphic section τ_∞ of L^k_T over $\mathcal{R}$ which satisfies

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}} |\tau_\infty|^2_{h^k_T} \leq C(R) \cdot r^{-2n} \cdot k^n \cdot \int_{B_{d_T}(x,2r)\cap\mathcal{R}} |\tau_\infty|^2_{h^k_T} \omega^n_T$$

whenever $B_{d_T}(x, 2r) \subset B_{d_T}(x_T, R)$.

4.4. **Gradient estimate to holomorphic sections.** In this subsection we introduce a family of Hermitian metrics on L which are

$$h_{FS,\epsilon} = h^{l_0}_{L'} \otimes \left(\frac{\Omega}{(\epsilon^2 + |s_D|^2_{h_D})^{1-\beta}}\right)^{-l_0 T} \otimes h_D^{l_0 T(1-\beta)} \cdot e^{-l_0 T(1-\beta)\log(\epsilon^2+|s_D|^2_{h_D})}$$

The metric $h_{FS,\epsilon}$ has curvature

$$\Theta_{h_{FS,\epsilon}} = l_0 \eta_T.$$

where η_T is the induced Fubini-Study metric which satisfies $\eta_T \leq C\omega_{t,\epsilon}$ for some C independent of t and ϵ; see Section 3. An easy calculation shows that for any $t \in [T - \bar{t}, T)$,

$$h_{t,\epsilon} = e^{-l_0 \frac{T}{t} u_{t,\epsilon}} h_{FS,\epsilon},$$

so $h_{t,\epsilon}$ is uniformly equivalent to $h_{FS,\epsilon}$.

In the following computation we denote $\nabla\tau = \nabla^{h^k_{FS,\epsilon}}\tau$, $\nabla\bar{\nabla}\tau = \nabla^{h^k_{FS,\epsilon}}\bar{\nabla}^{h^k_{FS,\epsilon}}\tau$, and $|\nabla\tau| = |\nabla^{h^k_{FS,\epsilon}}\tau|_{h^k_{FS,\epsilon}\otimes\omega_{t,\epsilon}}$, etc., for any $\tau \in H^0(M, L^k)$, $k \geq 1$.

Lemma 4.16. *For any $t \in [T - \bar{t}, T)$, $\epsilon > 0$ and $\tau \in H^0(M, L^k)$, $k \geq 1$, one has*

$$\triangle|\tau|^2 \geq |\nabla\tau|^2 - Ck|\tau|^2$$

and

$$\triangle|\nabla\tau|^2 \geq |\nabla\nabla\tau|^2 + |\nabla\bar{\nabla}\tau|^2 - kl_0\nabla_j(\eta_T)_{i\bar{j}}\langle\tau, \nabla_{\bar{i}}\bar{\tau}\rangle - kl_0\nabla_{\bar{j}}(tr_{\omega_{t,\epsilon}}\eta_T)\langle\nabla_j\tau, \bar{\tau}\rangle - Ck|\nabla\tau|^2.$$

Proof. They are direct consequences of the Bochner formulas (Lemma (4.1)) and $Ric(\omega_{t,\epsilon}) \geq -\frac{1}{t}\omega_{t,\epsilon}$. □

Proposition 4.17. *For any $R > 0$, there exists $C(R)$ independent of t, ϵ and $k \geq 1$ such that for any $t \in [T - \bar{t}, T)$ and $B_{\omega_{t,\epsilon}}(x, 2r) \subset B_{\omega_{t,\epsilon}}(x_0, R)$, if $\tau \in H^0(B_{\omega_{t,\epsilon}}(x, 2r), L^k)$, then*

$$\sup_{B_{\omega_{t,\epsilon}}(x,r)} |\tau|^2_{h^k_{FS,\epsilon}} \leq C(R) \cdot r^{-2n} \cdot k^n \cdot \int_{B_{\omega_{t,\epsilon}}(x,2r)} |\tau|^2_{h^k_{FS,\epsilon}} \omega^n_{t,\epsilon}$$

and

$$\sup_{B_{\omega_{t,\epsilon}}(x,r)} |\nabla^{h^k_{FS,\epsilon}}\tau|^2_{h^k_{FS,\epsilon}\otimes\omega_{t,\epsilon}} \leq C(R) \cdot r^{-2n-2} \cdot k^{n+1} \cdot \int_{B_{\omega_{t,\epsilon}}(x,2r)} |\tau|^2_{h^k_{FS,\epsilon}} \omega^n_{t,\epsilon}.$$

The proof of this proposition need to use Lemma (4.17) and Nash-Moser iteration. Because its proof is exactly same as Proposition (3.17) in [17], we omit it.

In subsection (4.3) we construct a Hermitian line bundle (L_T, h_T) on $\mathcal{R}$. Notice that $h_T = e^{-l_0 u_T} h_{FS,\epsilon} = e^{-l_0 u_T} \cdot h_{L'}^{l_0} \otimes \Omega^{-l_0 T} \otimes h_D^{l_0 T(1-\beta)}$. The following lemma is very useful(c.f. Lemma (3.19) [17]).

Lemma 4.18. *There is a family of cut-off functions $\gamma_\kappa \in C_0^\infty(\mathcal{R})$, $\kappa > 0$, with $0 \le \gamma_\kappa \le 1$ such that $\gamma_\kappa^{-1}(1)$ forms an exhaustion of $\mathcal{R}$ and, moreover,*

$$\int_{M_T} |\overline{\partial}\gamma_\kappa|^2 \omega_T^n \to 0, \ \ as\ \kappa \to 0.$$

By a standard iteration we have(c.f.[20])

Proposition 4.19. *Let $R > 0$, $t_i \to T$, $\epsilon \to 0$ and τ_i be a sequence of holomorphic sections of L^k, $k \ge 1$, satisfying*

$$\int_M |\tau_i|^2_{h^k_{h_{t_i,\epsilon_i}}} \omega^n_{t_i,\epsilon_i} \le 1.$$

Then, passing to a subsequence if necessary, τ_i converges to a locally bounded holomorphic section τ_∞ of L_T^k over $\mathcal{R}$ which satisfies

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}} |\nabla^{h_T^k}\tau_\infty|^2_{h_T^k\otimes\omega_T} \le C(R)\cdot r^{-2n-2}\cdot k^{n+1}\cdot \int_{B_{d_T}(x,2r)\cap\mathcal{R}} |\tau_\infty|^2_{h_T^k}\omega_T^n.$$

whenever $B_{d_T}(x, 2r) \subset B_{d_T}(x_T, R)$.

4.5. **Algebraic structure of M_T.** Recall that if $\tau \in H^0(M, L^k)$, then by the construction of (L_T, h_T) on $\mathcal{R}$, one knows that $\tau|_\mathcal{R}$ denoted by τ_∞ is a holomorphic section of (L_T, h_T). For a fixed $\tau \in H^0(M, L^k)$, one has

$$\int_M |\tau|^2_{h_{t,\epsilon}}\omega^n_{t,\epsilon} \le C\int_M |\tau|^2_{h_{FS,\epsilon}} \frac{\Omega}{|s_D|^{2(1-\beta)}_{h_D}} \le C_\tau.$$

Therefore, by $h_{t,\epsilon} \xrightarrow{C^\infty} h_T$ and Lemma (4.14), we have

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}} |\tau_\infty|^2_{h_T^k} \le C(R, r, k)$$

Notice that

$$\begin{aligned} |\nabla^{h_T^k}\tau_\infty|_{h_T^k\otimes\omega_T} &\le |\nabla^{h^k_{FS,\epsilon}}\tau_\infty|_{h_T^k\otimes\omega_T} + kl_0|\tau_\infty|_{h_T^k}\cdot|\nabla u_T|_{\omega_T} \\ &\le C|\nabla^{h^k_{FS,\epsilon}}\tau_\infty|_{h^k_{FS,\epsilon}\otimes\eta_T} + kl_0|\tau_\infty|_{h_T^k}\cdot|\nabla u_T|_{\omega_T}. \end{aligned}$$

where the last inequality base on the estimate $\omega_T \ge C^{-1}\eta_T$ and the fact that h_T is equivalent to $h_{FS,\epsilon}$.

From Proposition (3.9), $|\tau_\infty|_{h_T^k} \cdot |\nabla u_T|_{\omega_T}$ is bounded on $B_{d_T}(x,r) \cap \mathcal{R}$. By the Song's argument(Lemma (3.10) in [20]) one gets $|\nabla^{h^k_{FS,\epsilon}} \tau_\infty|_{h^k_{FS,\epsilon} \otimes \eta_T}$ is also bounded on $B_{d_T}(x,r) \cap \mathcal{R}$. Thus

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}} |\nabla^{h_T^k} \tau_\infty|_{h_T^k \otimes \omega_T} \leq C(R,r,k),$$

i.e. τ_∞ with metric h_T is locally Lipschitz, moreover it can be continuously extended to M_T.

So, the map

$$\Phi_T : \ (\mathcal{R}, d_T) \to (\Phi(M), \omega_{FS})$$

defined by Φ can be continuously extended to

$$\Phi_T : \ (M_T, d_T) \to (\Phi(M), \omega_{FS})$$

that is a Lipschitz map, since $\Phi_T^* \omega_{FS} = k l_0 \eta_T \leq C k l_0 \omega_T$.

Proposition 4.20. *Φ_T is injective and is a local homeomorphism.*

The proof of this Proposition is exactly same as Proposition (3.21) and Proposition (3.22) in [17] so we omit it.

5. Diameter bound of the conical Kähler metric

Let ω_T be the solution to the following equation in the current sense

$$(\omega_T + \sqrt{-1}\partial\bar\partial u_T)^n = e^{\frac{u_T}{T}} \frac{\Omega}{|s_D|_{h_D}^{2(1-\beta)}}.$$

In [20],Song developed a method to prove the diameter bound of a singular Kähler-Einstein metric. In this subsection, we follow his idea to show the diameter bound of $(M \backslash (D \cup \bar D), \omega_T)$ where $\bar D$ is any divisor such that $[\omega_0] - T c_1(M) + T(1-\beta) c_1(L_D) - \mu c_1(L_{\bar D}) > 0$ for some $\mu > 0$. We will consider the following three cases.

Case 1. If $p \in D \backslash \bar D$, then by Theorem (1.4) there exists a neighborhood U of p such that

$$\omega_T \leq C_U \frac{\omega_0}{|s_D|_{h_D}^{2(1-\beta)}}, \ \textit{on } U$$

.

Case 2.

Let $p \in D \cap \bar D$ be any point, $\pi : \ \widetilde{M} \to M$ be the blow-up at p with exceptional divisor $\pi^{-1}(p) = E$. Then

$$K_{\widetilde{M}} = \pi^* K_M + (n-1)E.$$

Let h_E be the Hermitian metric on L_E associated with the divisor E, and σ_E be a defining section. We denote by $D_1 = \overline{\pi^{-1}(D) - E}$, $h_{D_1} = \pi^* h_D$ and $D_2 = \overline{\pi^{-1}(\bar D) - E}$, $h_{D_2} = \pi^* h_{\bar D}$. Let χ be a fixed Kähler metric on $\widetilde{M}$. Let σ_{D_1} be a defining section on L_{D_1} and σ_{D_2} be a defining section on L_{D_2}.By the calculation one has

$$\pi^* \eta_T + \mu\sqrt{-1}\partial\bar\partial \log |\sigma_{D_2}|^2_{h_{D_2}} + \delta_0 \sqrt{-1}\partial\bar\partial \log |\sigma_E|^2_{h_E} \geq \delta_1 \chi$$

for some small $\delta_0, \delta_1 > 0$ on $\widetilde{M}\backslash(D_2 \cup E)$. Observe that $\tilde{\Omega} = |\sigma_E|_{h_E}^{-2(n-1)}\pi^*\Omega$ defines a smooth volume form on $\widetilde{M}$. Consider the following family of Monge-Ampère equations on $\widetilde{M}$

$$(\pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi_{\epsilon,\delta}})^n = e^{\frac{1}{T}\widetilde{\varphi_{\epsilon,\delta}}}(\epsilon^2 + |\sigma_E|_{h_E}^2)^{n-1}\frac{\tilde{\Omega}}{(\delta^2 + |\sigma_{D_1}|_{h_{D_1}}^2)^{1-\beta}}. \tag{5.1}$$

By Yau's solution to Calabi conjecture [33], the equation has a unique smooth solution $\widetilde{\varphi_{\epsilon,\delta}}$; moreover

$$\widetilde{\omega_{\epsilon,\delta}} = \pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi_{\epsilon,\delta}}$$

is a smooth Kähler metric on $\widetilde{M}$.

Lemma 5.2. *For any $\mu > 0$ and $\delta_0 > 0$, there exist $C(\mu, \delta_0)$ and C independent of ϵ and δ such that*

$$\mu\sqrt{-1}\partial\bar{\partial}\log|\sigma_{D_2}|_{h_{D_2}}^2 + \delta_0\sqrt{-1}\partial\bar{\partial}\log|\sigma_E|_{h_E}^2 - C(\mu, \delta_0) \le \widetilde{\varphi_{\epsilon,\delta}} \le C.$$

Proof. We follow Song's argument [21]. For upper bound, let

$$V_{\epsilon,\delta} = \int_{\widetilde{M}}(\epsilon^2 + |\sigma_E|_{h_E}^2)^{n-1}\frac{\tilde{\Omega}}{(\delta^2 + |\sigma_{D_1}|_{h_{D_1}}^2)^{1-\beta}}$$

be the volume. We see that

$$V_{1,0} \ge V_{\epsilon,\delta} \ge V_{0,1} = \int_{\widetilde{M}}\frac{\tilde{\Omega}}{|\sigma_{D_1}|_{h_{D_1}}^{2(1-\beta)}}$$

hence $V_{\epsilon,\delta}$ is uniformly bounded. We denote $\widetilde{\Omega_{\epsilon,\delta}} = (\epsilon^2 + |\sigma_E|_{h_E}^2)^{n-1}\frac{\tilde{\Omega}}{(\delta^2+|\sigma_{D_1}|_{h_{D_1}}^2)^{1-\beta}}$, then we have the following calculation

$$\begin{aligned}\frac{1}{V_{\epsilon,\delta}}\int_{\widetilde{M}}\frac{1}{T}\widetilde{\varphi_{\epsilon,\delta}}\widetilde{\Omega_{\epsilon,\delta}} &= \frac{1}{V_{\epsilon,\delta}}\int_{\widetilde{M}}\log\left(\frac{\widetilde{\omega_{\epsilon,\delta}}^n}{\widetilde{\Omega_{\epsilon,\delta}}}\right)\cdot\widetilde{\Omega_{\epsilon,\delta}} \\ &\le \log\int_{\widetilde{M}}\widetilde{\omega_{\epsilon,\delta}}^n - \log V_{\epsilon,\delta} \\ &= \log(\int_{\widetilde{M}}(\pi^*\eta_T + \epsilon\chi)^n) - C \le C\end{aligned}$$

where for the first inequality we use Jensens inequality. Since $\widetilde{\varphi_{\epsilon,\delta}} \in PSH(\widetilde{M}, \pi^*\eta_T + \epsilon\chi)$,the mean value inequality implies that

$$\sup_{\widetilde{M}}\widetilde{\varphi_{\epsilon,\delta}} \le C.$$

For the lower bound, we set $\widetilde{\varphi_{\epsilon,\delta}}' = \widetilde{\varphi_{\epsilon,\delta}} - \mu\sqrt{-1}\partial\bar{\partial}\log|\sigma_{D_2}|^2_{h_{D_2}} - \delta_0\sqrt{-1}\partial\bar{\partial}\log|\sigma_E|^2_{h_E}$, then by (5.1) one knows

$$((\pi^*\eta_T + \mu\sqrt{-1}\partial\bar{\partial}\log|\sigma_{D_2}|^2_{h_{D_2}} + \delta_0\sqrt{-1}\partial\bar{\partial}\log|\sigma_E|^2_{h_E}) + \epsilon\chi + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi_{\epsilon,\delta}}')^n = e^{\frac{1}{T}\widetilde{\varphi_{\epsilon,\delta}}'} \cdot \frac{1}{T}|\sigma_{D_2}|^{2\mu}_{h_{D_2}} \cdot \frac{1}{T}|\sigma_E|^{2\delta_0}_{h_E} \cdot (|\sigma_E|^2_{h_E} + \epsilon^2)^{n-1} \cdot \frac{\tilde{\Omega}}{(|\sigma_{D_1}|^2_{h_{D_1}} + \delta^2)^{1-\beta}}. \tag{5.3}$$

We consider the following Monge-Ampére equations

$$(\pi^*\eta_T - \mu Ric(h_{D_2}) - \delta_0 Ric(h_E) + \epsilon\chi + \sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta})^n = e^{\frac{1}{T}\psi_{\epsilon,\delta}} \cdot (|\sigma_E|^2_{h_E} + \epsilon^2)^{n-1} \cdot \frac{\tilde{\Omega}}{(|\sigma_{D_1}|^2_{h_{D_1}} + \delta^2)^{1-\beta}}.$$

By Yau's theorem [33], the above equation admits a unique smooth solution. By [9], we have

$$|\psi_{\epsilon,\delta}|_{C^0} \le C(\mu, \delta_0).$$

Set $H_{\epsilon,\delta} = \widetilde{\varphi_{\epsilon,\delta}}' - \psi_{\epsilon,\delta}$ and $\nu_\epsilon = \pi^*\eta_T - \mu Ric(h_{D_2}) - \delta_0 Ric(h_E) + \epsilon\chi$, then on $\widetilde{M}\backslash(E \cup D_1 \cup D_2)$ one knows

$$\log\frac{(\nu_\epsilon + \sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta} + \sqrt{-1}\partial\bar{\partial}H_{\epsilon,\delta})^n}{(\nu_\epsilon + \sqrt{-1}\partial\bar{\partial}\psi_{\epsilon,\delta})^n} = \frac{1}{T}H_{\epsilon,\delta} - 2\log T + \log|\sigma_{D_2}|^{2\mu}_{h_{D_2}} + \log|\sigma_E|^{2\delta_0}_{h_E}.$$

The minimum of $H_{\epsilon,\delta}$ cannot be at $D_2 \cup E$. Assume $H_{\epsilon,\delta}$ attains minimum at x_0, then by maximum principle, one gets

$$(\frac{1}{T}H_{\epsilon,\delta} - 2\log T + \log|\sigma_{D_2}|^{2\mu}_{h_{D_2}} + \log|\sigma_E|^{2\delta_0}_{h_E})(x_0) \ge 0.$$

Hence we know

$$\inf_{\widetilde{M}} H_{\epsilon,\delta} \ge -C.$$

By the C^0 estimate of $\psi_{\epsilon,\delta}$, we obtain the lower bound of $\widetilde{\varphi_{\epsilon,\delta}}$. □

Lemma 5.4. *There exist C and λ_1 independent of ϵ and δ such that*

$$Ric(\widetilde{\omega_{\epsilon,\delta}}) \le -\frac{1}{T}\widetilde{\omega_{\epsilon,\delta}} + C\frac{\chi}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}}}.$$

Proof. First, we observe following facts:

(1) Since $\tilde{\Omega}$ is a smooth volume form, $Ric(\tilde{\Omega}) \le C\chi$.

(2) $\sqrt{-1}\partial\bar{\partial}\log(|\sigma_E|^2_{h_E} + \epsilon^2)^{n-1} \ge -C\chi$.

(3) $\pi^*\eta_T \le C\chi$.

(4) If λ_1 is sufficiently large, one has $\sqrt{-1}\partial\bar{\partial}\log(|\sigma_{D_1}|^2_{h_{D_1}} + \delta^2) \le \frac{C\chi}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}}}$.

Thus by a simple calculation one gets

$$Ric(\widetilde{\omega_{\epsilon,\delta}}) \leq -\frac{1}{T}\widetilde{\omega_{\epsilon,\delta}} + C\frac{\chi}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}}}.$$

□

Lemma 5.5. *There exist C and λ independent of ϵ and δ such that*

$$\widetilde{\omega_{\epsilon,\delta}} \leq \frac{C}{|\sigma_E|^{2\lambda}_{h_E}|\sigma_{D_1}|^{2\lambda}_{h_{D_1}}|\sigma_{D_2}|^{2\lambda}_{h_{D_2}}}\chi.$$

Proof. By a standard calculation one has

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \log tr_{\chi}\widetilde{\omega_{\epsilon,\delta}} \geq -Ctr_{\widetilde{\omega_{\epsilon,\delta}}}\chi - \frac{C}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}}}.$$

There is a easy fact that is

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}}\widetilde{\varphi_{\epsilon,\delta}} = n - tr_{\widetilde{\omega_{\epsilon,\delta}}}\pi^*\eta_T - \epsilon tr_{\widetilde{\omega_{\epsilon,\delta}}}\chi.$$

Let $H = \log(|\sigma_E|^{2A}_{h_E}|\sigma_{D_1}|^{2A}_{h_{D_1}}|\sigma_{D_2}|^{2A}_{h_{D_2}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}}) - A^2\widetilde{\varphi_{\epsilon,\delta}}$. Then, on $\tilde{M}\backslash(E\cup D_1\cup D_2)$, we get

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}}H \geq -Ctr_{\widetilde{\omega_{\epsilon,\delta}}}\chi - \frac{C}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}}} - A^2 n + Atr_{\widetilde{\omega_{\epsilon,\delta}}}(A\pi^*\eta_T - Ric(h_E) - Ric(h_{D_1}) - Ric(h_{D_2})).$$

Notice that when A is sufficiently large we observe that

$$Atr_{\widetilde{\omega_{\epsilon,\delta}}}(A\pi^*\eta_T - Ric(h_E) - Ric(h_{D_1}) - Ric(h_{D_2})) \geq (C+1)tr_{\widetilde{\omega_{\epsilon,\delta}}}\chi.$$

Therefore

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}}H \geq tr_{\widetilde{\omega_{\epsilon,\delta}}}\chi - \frac{C}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}}} - A^2 n.$$

Assume that H attains maximum at x_0 ($x_0 \in \widetilde{M}\backslash(E\cup D_1\cup D_2)$), one deduces

$$(|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}})(tr_{\widetilde{\omega_{\epsilon,\delta}}}\chi - A^2 n)(x_0) \leq C$$

Using an inequality $\widetilde{\omega_{\epsilon,\delta}}^n \leq C\frac{\chi^n}{|\sigma_{D_1}|^{2(1-\beta)}_{h_{D_1}}}$, one obtains

$$\frac{1}{C}|\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}}(tr_{\chi}\widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}} \leq tr_{\widetilde{\omega_{\epsilon,\delta}}}\chi$$

Thus

$$(|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi}\widetilde{\omega_{\epsilon,\delta}})(\frac{1}{C}|\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}}(tr_{\chi}\widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}} - A^2 n)(x_0) \leq C. \tag{5.6}$$

If

$$(tr_{\chi}\widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}}(x_0) \leq \frac{2CA^2 n}{|\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}}}(x_0),$$

then there exists λ_2 such that

$$tr_\chi \widetilde{\omega_{\epsilon,\delta}}(x_0) \le \frac{C}{|\sigma_{D_1}|^{2\lambda_2}_{h_{D_1}}}(x_0).$$

Otherwise,

$$(tr_\chi \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}}(x_0) \ge \frac{2CA^2 n}{|\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}}}(x_0),$$

from (5.6) one knows that

$$tr_\chi \widetilde{\omega_{\epsilon,\delta}}(x_0) \le \frac{C}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}}}(x_0).$$

In general one can find λ' such that

$$tr_\chi \widetilde{\omega_{\epsilon,\delta}}(x_0) \le \frac{C}{|\sigma_{D_1}|^{2\lambda'}_{h_{D_1}}}(x_0).$$

Choose $A >> \lambda'$, one knows $H \le C$. Therefore the Lemma is proved. □

Let B be a disk centered at p and $\tilde{B} = \pi^{-1}(B)$. Denote $f_1, \cdots, f_N$ as the defining functions of divisors D_1 and D_2.

Corollary 5.7. *There exist C and λ independent of ϵ and δ such that*

$$\widetilde{\omega_{\epsilon,\delta}}|_{\partial\tilde{B}} \le C(\prod_{i=1}^{N} |f_i|^{-2\lambda}\chi)|_{\partial\tilde{B}}.$$

Let $\hat{\chi}$ be the smooth closed nonnegative closed $(1,1)$-form as the pullback of the Euclidean metric $\sqrt{-1}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$ on B. $\hat{\chi}$ is a kähler metric on $\tilde{B}\backslash E$.

Lemma 5.8. *There exists $C > 0$, a sufficiently small $\epsilon_0 > 0$ and a smooth Hermitian metric h_E on L_E such that in $\tilde{B}$*

$$C^{-1}\hat{\chi} \le \chi \le C\frac{\hat{\chi}}{|\sigma_E|^2_{h_E}},$$
$$\pi^*\eta_T - \epsilon_0 Ric(h_E) > 0.$$

The following proposition is the main result of this section.

Proposition 5.9. *There exist $0 < \alpha < 1$, λ and $C > 0$ independent of ϵ and δ such that*

$$\widetilde{\omega_{\epsilon,\delta}} \le \frac{C}{|\sigma_E|^{2(1-\alpha)}_{h_E}\prod_{i=1}^N |f_i|^{2\lambda}}\chi, \ \textit{in } \tilde{B}.$$

Proof. Let $H_{\epsilon,\delta} = \log(|\sigma_E|^{2(1+r)}_{h_E} \cdot \prod_{i=1}^N |f_i|^{2\lambda} \cdot tr_{\hat{\chi}}\widetilde{\omega_{\epsilon,\delta}}) - A\widetilde{\varphi_{\epsilon,\delta}}$ for some sufficiently large A and sufficiently small r. There are some facts in $\tilde{B}\backslash(E \cup D_1 \cup D_2)$:

(1) $\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \log|\sigma_E|^2_{h_E} = -tr_{\widetilde{\omega_{\epsilon,\delta}}}(Ric(h_E))$,

(2) $\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \log \prod_{i=1}^{N} |f_i|^{2\lambda} = 0$,

(3) $\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \widetilde{\varphi_{\epsilon,\delta}} = n - tr_{\widetilde{\omega_{\epsilon,\delta}}} \pi^* \eta_T - \epsilon tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi$,

(4) $\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \log tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}} \geq -C tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - C(|\sigma_E|^2_{h_E} |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}})^{-1}$.

Thus in $\tilde{B} \backslash (E \cup D_1 \cup D_2)$ one has

$$\begin{aligned}
\triangle_{\widetilde{\omega_{\epsilon,\delta}}} H_{\epsilon,\delta} &\geq -C tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - \frac{C}{|\sigma_E|^2_{h_E} |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}} - An - (r+1) tr_{\widetilde{\omega_{\epsilon,\delta}}} (Ric(h_E)) + A tr_{\widetilde{\omega_{\epsilon,\delta}}} \pi^* \eta_T \\
&\geq tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - \frac{C}{|\sigma_E|^2_{h_E} |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}} - An
\end{aligned}$$

where the last inequality base on Lemma (5.8) and the sufficiently large number A.

By a similar calculation one gets

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}} \log tr_{\chi} \widetilde{\omega_{\epsilon,\delta}} \geq -C_1 tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - \frac{C_1}{|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi} \widetilde{\omega_{\epsilon,\delta}}}.$$

Let $G_{\epsilon,\delta} = H_{\epsilon,\delta} + \frac{1}{2C_1} \log \prod_{i=1}^{N} |f_i|^{2\lambda+2} tr_{\chi} \widetilde{\omega_{\epsilon,\delta}}$. By the same argument one knows

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}} G_{\epsilon,\delta} \geq \frac{1}{2} tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - An - \frac{1}{2|\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\chi} \widetilde{\omega_{\epsilon,\delta}}} - \frac{C}{|\sigma_E|^2_{h_E} |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}}.$$

By Lemma (5.8) and the above inequality, we have

$$\triangle_{\widetilde{\omega_{\epsilon,\delta}}} G_{\epsilon,\delta} \geq \frac{1}{2} tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - An - \frac{C}{|\sigma_E|^2_{h_E} |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}}.$$

For fixed sufficiently large $\lambda > 0$, there exists $C > 0$ such that

$$\sup_{\partial \tilde{B}} G_{\epsilon,\delta} \leq C$$

from the estimate in Corollary (5.7).

So we assume that

$$\sup_{\tilde{B}} G_{\epsilon,\delta} = G_{\epsilon,\delta}(p_{max})$$

for some $p_{max} \in \tilde{B} \backslash (E \cup D_1 \cup D_2)$. Then at p_{max}

$$(tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi - 2An) \cdot |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} |\sigma_E|^2_{h_E} \cdot tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}(p_{max}) \leq C.$$

Notice that

$$\frac{1}{C} |\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}} (tr_{\chi} \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}} \leq tr_{\widetilde{\omega_{\epsilon,\delta}}} \chi.$$

Then we have

$$\left(\frac{1}{C} |\sigma_{D_1}|^{\frac{2(1-\beta)}{n-1}}_{h_{D_1}} (tr_{\chi} \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}} - 2An\right) \cdot |\sigma_{D_1}|^{2\lambda_1}_{h_{D_1}} |\sigma_E|^2_{h_E} \cdot tr_{\hat{\chi}} \widetilde{\omega_{\epsilon,\delta}}(p_{max}) \leq C. \tag{5.10}$$

If

$$(tr_\chi \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}}(p_{max}) \le \frac{3CAn}{|\sigma_{D_1}|_{h_{D_1}}^{\frac{2(1-\beta)}{n-1}}}(p_{max}),$$

then we observe that

$$tr_{\hat\chi}\widetilde{\omega_{\epsilon,\delta}}(p_{max}) \le \frac{C}{|\sigma_E|^2_{h_E}|\sigma_{D_1}|^{2(1-\beta)}_{h_{D_1}}}(p_{max}).$$

Hence $G_{\epsilon,\delta}$ is bounded above by a uniform constant.

Otherwise

$$(tr_\chi \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}}(p_{max}) \ge \frac{3CAn}{|\sigma_{D_1}|_{h_{D_1}}^{\frac{2(1-\beta)}{n-1}}}(p_{max}),$$

i.e.

$$An \le \frac{1}{3C}|\sigma_{D_1}|_{h_{D_1}}^{\frac{2(1-\beta)}{n-1}} \cdot (tr_\chi \widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{n-1}}(p_{max})$$

Then by (5.10) one gets

$$\log|\sigma_E|^2_{h_E} + \log tr_{\hat\chi}\widetilde{\omega_{\epsilon,\delta}} + \frac{1}{n-1}\log tr_\chi\widetilde{\omega_{\epsilon,\delta}} + \log|\sigma_{D_1}|_{h_{D_1}}^{2\lambda_1+\frac{2(1-\beta)}{n-1}}(p_{max}) \le C.$$

Moreover, combining with the Lemma (5.2) and choosing sufficiently large λ one knows

$$G_{\epsilon,\delta}(p_{max}) \le C$$

In sun, in all cases, we have $G_{\epsilon,\delta} \le C$. Then

$$\log(|\sigma_E|^{2(1+r)}_{h_E}\prod_{i=1}^N |f_i|^{2\lambda+\frac{2\lambda+2}{2C_1}} \cdot (tr_{\hat\chi}\widetilde{\omega_{\epsilon,\delta}}) \cdot (tr_\chi\widetilde{\omega_{\epsilon,\delta}})^{\frac{1}{2C_1}}) \le C.$$

Noting that $tr_{\hat\chi}\widetilde{\omega_{\epsilon,\delta}} \ge C^{-1} tr_\chi \widetilde{\omega_{\epsilon,\delta}}$, we have

$$(tr_\chi\widetilde{\omega_{\epsilon,\delta}})^{1+\frac{1}{2C_1}} \le \frac{C}{|\sigma_E|^{2(1+r)}_{h_E}\prod_{i=1}^N|f_i|^{2\Lambda}}.$$

If we choose $r = \frac{1}{10C_1}$, then $\frac{1+r}{1+(2C_1)^{-1}} = 1-\alpha$ for some $\alpha \in (0,1)$. The Proposition is proved. □

Corollary 5.11. *Assume as above. There exist $\alpha > 0$, $\lambda > 0$ and $C > 0$ such that*

$$\pi^*\omega_T \le \frac{C}{|\sigma_E|^{2(1-\alpha)}_{h_E}\prod_{i=1}^N|f_i|^{2\lambda}}\chi, \ in\ \tilde B.$$

Case 3.

Let $p \in \bar D\backslash D$ be any point and π, $\widetilde{M}$, E, h_E, σ_E, D_2, h_{D_2}, σ_{D_2}, χ and $\tilde\Omega$ be the same as **Case 2**. Consider the following family of Monge-Ampère equations on $\widetilde{M}$

$$(\pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar\partial\widetilde{\varphi_{\epsilon,\delta}})^n = e^{\frac{1}{T}\widetilde{\varphi_{\epsilon,\delta}}}(\epsilon^2 + |\sigma_E|^2_{h_E})^{n-1}\frac{\tilde\Omega}{(\delta^2+|s_D|^2_{h_D})^{1-\beta}}.$$

By Yau's solution to Calabi conjecture [33], the equation has a unique smooth solution $\widetilde{\varphi_{\epsilon,\delta}}$; moreover

$$\widetilde{\omega_{\epsilon,\delta}} = \pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar{\partial}\widetilde{\varphi_{\epsilon,\delta}}$$

is a smooth Kähler metric on $\widetilde{M}$.

Let B be a disk centered at p such that $B \cap D = \emptyset$ and $\tilde{B} = \pi^{-1}(B)$. Denote $f_1, \cdots, f_{N_1}$ as the defining functions of divisor D_2. By the same argument of Proposition (5.9) we have

Corollary 5.12. *There exist $0 < \alpha < 1$, λ and $C > 0$ independent of ϵ and δ such that*

$$\pi^*\omega_T \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\alpha)} \prod_{i=1}^{N_1} |f_i|^{2\lambda}}\chi, \text{ in } \tilde{B}.$$

From now on we turn to the Gromov-Hausdorff convergence. By the argument of [17], Corollary (5.11) and Corollary (5.12), we immediately conclude the following proposition.

Proposition 5.13. $\Phi_T :\ M_T \to \Phi(M)$ *s a homeomorphism. As a consequence, the diameter of M_T is finite. Furthermore, there exists C such that*

$$diam(M, \omega_t) \leq C, \ \forall\ t \in [T - \bar{t}, T).$$

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Yan Li
School of Mathematical Sciences,
Capital Normal University,
105 Xisanhuanbeilu,
100048 Beijing, China
E-mail address: liyandota@hotmail.com